$\mathbb{Z}_5$

0, 1, 2, 3, 4, 5

Lecture 7: Nowhere-zero integer flows

A *weight* of a graph G is a function $f : E(G) \to \Gamma$ that assigns to every edge an element of an Abelian group Γ. An *orientation* of a graph is assignment to each edge one of the two possible directions. By assigning an orientation D of a graph G, we obtain a directed graph, which we denote by $D(G)$. We will treat an orientation D as a function, for which the following holds:

DIRECTED GRAPH

$D(u,v) = 1$ $\quad D(v,u) = -1$

$$D(u,v) = \begin{cases} 1, & \text{orientation of the edge } uv \text{ is from } u \text{ to } v \\ -1, & \text{otherwise.} \end{cases}$$

Now observe that for every edge uv, it holds $D(u,v) = -D(v,u)$. $= -1$

A Γ-*flow* or just a *flow* of graph G is ordered pair (D, f), where D is an orientation and f is a weight of the graph G, which satisfy the *Kirchhoff law*:

NO SOURCE AND SINK (CIRCULATION)

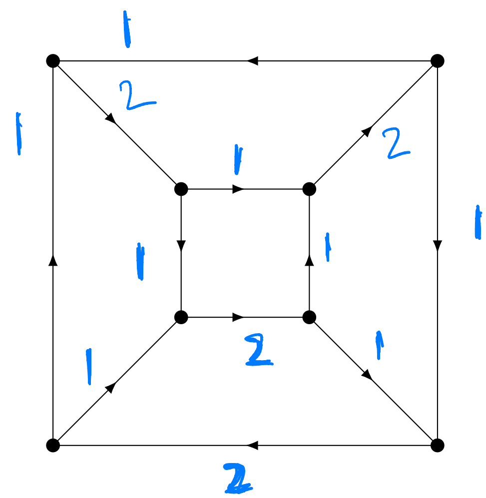
$$\forall v \in V(G) : \sum_{u \in N(v)} D(v,u) f(vu) = 0, \tag{1}$$

where $N(v)$ denotes the set of neighbours of v.

$-2-3+1+2+2=0$

1: Find a flow using $\mathbb{Z}$. Can you find a flow, where 0 is not used and all values are positive?

$\mathbb{Z}_5$



$-3-2-2+2=0$

$2+3+3+2=0$

G-FLOW

3-FLOW

For a weight f of G, the *support* is the set of edges $e \in E(G)$, for which $f(e) \neq 0$. Usually, we denote the support by $\mathrm{supp}(f)$. A flow (D, f) of a graph G is *nowhere-zero*, if $\mathrm{supp}(f) = E(G)$. A flow (D, f) is

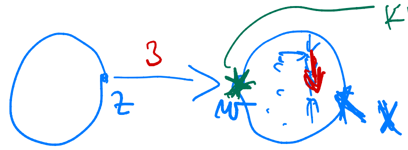
- an *integer* flow if f maps into $(\mathbb{Z}, +)$
- a k-*flow* if f maps into $\mathbb{Z}$ and $|f(e)| < k$ for every edge $e \in E(G)$.

If every edge of G has a positive weight of an integer flow f, then f is a *positive* flow. A *flow number* $\kappa(G)$ of a graph G, is the smallest number k, for which G admits nowhere-zero k-flow. If such k does not exists, then we define $\kappa(G) = \infty$.

$\xrightarrow{3}$

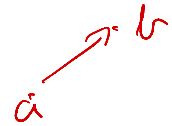
2: Prove that the following holds:

1. Graphs with bridges do not admit nowhere-zero flows.



$$\sum_{X \in X} \overbrace{\sum_{x,m} D(x,m)\, f(x,m)}^{0} = 0$$

$D(w,z)\, f(z,w)$ $D(a,b)\, f(a,b) + D(b,a)\, f(b,a)$



2. If a graph admits a nowhere-zero k-flow, then for every $h \geq k$, it admits a nowhere-zero h-flow.

(D, f) IS K-FLOW $\Rightarrow$ IT IS h-FLOW IF $h \geq K$

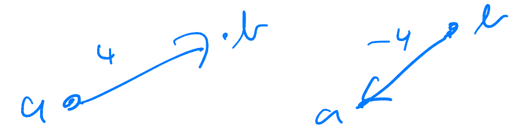
3. Let (D, f) a (nowhere-zero) flow of a graph G and let $F \subseteq E(G)$ be some subset of edges of G. Let D_F be the orientation, which we get from D by changing the orientations of all edges of F. Define a weight f_F of G in the following way:

$$f_F(e) = \begin{cases} f(e), & e \notin F \\ -f(e), & e \in F. \end{cases}$$

Then (D_F, f_F) is also (nowhere-zero) flow of G.

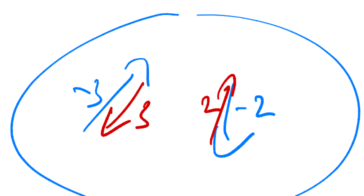
$D(a,b)\, f(a,b) = -D(a,b) \cdot -f(a,b) = D_F(a,b)\, f_F(a,b)$

IF a,b THE ONLY EDGE IN F

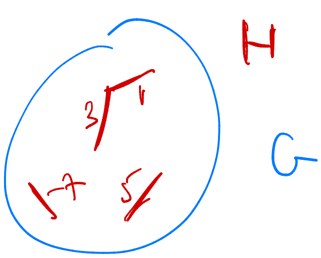


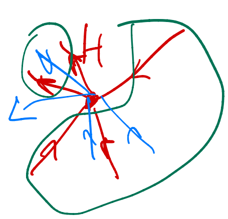
4. If a graph G admits a nowhere-zero Γ-flow (k-flow) for a given orientation, then it also admits a nowhere-zero Γ-flow (k-flow) for any orientation. In particular, if a graph admits nowhere-zero k-flow, then it also admits a positive k-flow.

F .. ALL EDGES WITH NEGATIVE WEIGHT & USE 3 $F = \{e, f(e) < 0\}$



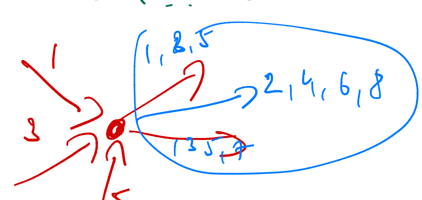
5. For a given integer flow of a graph G, let H be the subgraph of G induced by the edges of odd weights. Then, H is an even graph. In particular, from here it follows that a graph admits nowhere-zero 2-flow, if and only if it is an even graph.





ODD IN ... ODD SUM   NO CHANGE IN PARITY

EVEN OUT .. EVEN SUM



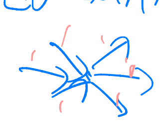
ONLY EVEN OUT

9 IN

OUT

2-FLOW .. WEIGHT 1 & -1

EULERIAN



$\mathbb{Z}_5$

Theorem 1 (Tutte). *A graph admits nowhere-zero k-flow if and only if it admits nowhere-zero $\mathbb{Z}_k$-flow.*

K-FLOW $\Rightarrow$ $\mathbb{Z}_\ell$-FLOW FOR $\ell \leq K$

ℓ-FLOW $\Leftarrow$

1 Flow polynomials

Recall $P(G,k)$ is a chromatic polynomial.

For a fixed orientation of G, $F(G,k)$, the number of different nowhere-zero Γ-flows is a polynomial of k, with $|\Gamma| = k$. It will also be a polynomial.

Proposition 2. *Function $F(G,k)$ has the following properties:*

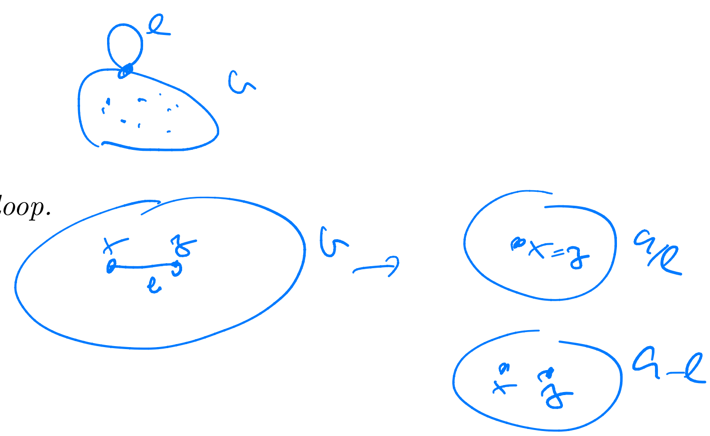
ALLOW MULTIGRAPH

(1) *$F(G,k) = 0$, if G is just an edge;*

(2) *$F(G,k) = k-1$, if G is just a loop;*

(3) *$F(G,k) = (k-1)F(G-e,k)$, if $e \in E(G)$ is a loop;*

(4) *$F(G,k) = F(G/e,k) - F(G-e,k)$, if $e \in E(G)$ is not a loop.*



3: Prove the proposition.

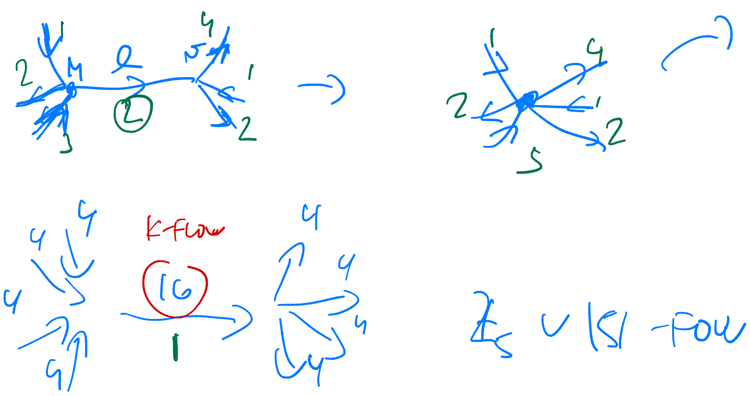
(1) $F(G,K) = 0$

(2) ← ANY EXCEPT 0 GIVES $(K-1)$ OPTIONS

(3) ALL IN $\Gamma - 0$

ANY FLOW IN $G-e$ EXTENDS TO $(K-1)$ FLOWS IN G

(4)



COPY f TO G

& USE e TO BALANCE u, v

IF $f(e) = 0$ THEN

SUBSTRACTED IN $F(G-e,K)$

$\mathbb{Z}_5$ v |5| -FLOW

From the above proposition, by induction on the number of edges of the graphs, it easily follows that $F(G,k)$ is a polynomial depending only of G and k (and not of Γ). This gives us the next two interesting consequences.

Corollary 3. *Let G be a graph with an arbitrary orientation D, and let Γ_1, Γ_2 be Abelian groups of order k. Then, the number of nowhere-zero Γ_1-flows of G is equal to the number of nowhere-zero Γ_2-flows of G.*

In particular, from the above one, we obtain the following consequence.

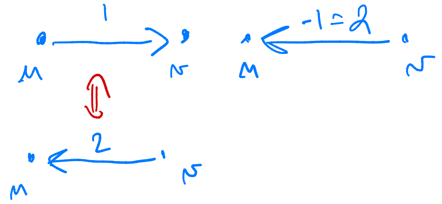
Corollary 4. *Let G be a graph and let Γ_1 and Γ_2 be Abelian groups of order k. Then, G admits nowhere-zero Γ_1-flow if and only if it admits nowhere-zero Γ_2-flow.*

K

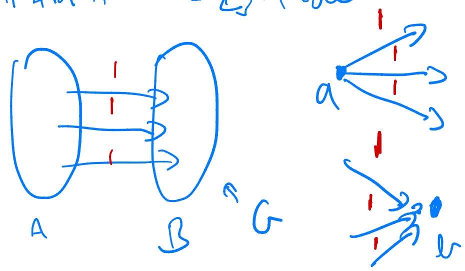
$1, 2$
$-1, 2 \in \mathbb{Z}_3 .. \{0, 1, 2\}$

Proposition 5. *A cubic graph admits a nowhere-zero 3-flow if and only if it is bipartite.* 2-COLORABLE

4: Prove the proposition. Hint: Try nowhere-zero $\mathbb{Z}_3$-flow. What happens when reversing an edge with its weight?



BIPARTITE $\Rightarrow \mathbb{Z}_3$-FLOW



IN $\mathbb{Z}_3$
Σ IS 0
FOR BOTH
a, b

$\mathbb{Z}_3$-FLOW $\Rightarrow$ BIPARTITE

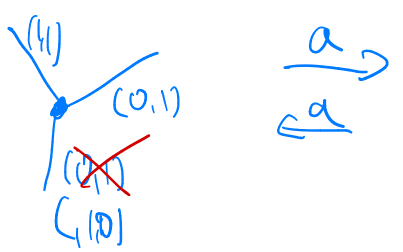
POSITIVE $\Leftarrow$ 3-FLOW

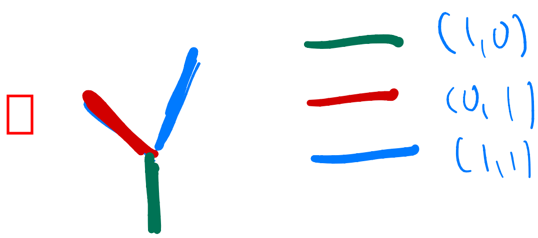
ALSO $\mathbb{Z}_3$ 1 ON EVERY EDGE

Proposition 6. *A cubic graph admits a nowhere-zero 4-flow if and only if it is 3-edge-colorable.*

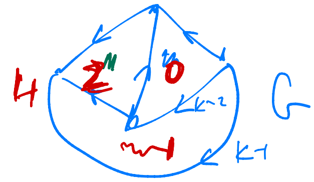
5: Prove the proposition: Hint: Try nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows, what values may appear on edges around 1 vertex?

$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$





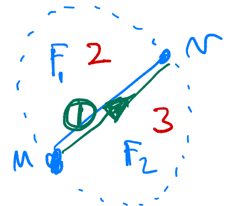
2 Flows and colorings



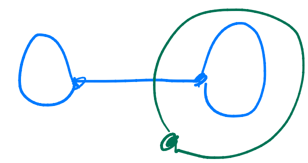
A nowhere-zero integer k-flow of a plane graph induces k-coloring of the dual graph, and vice versa. So somehow it turns that the theory of flows is a natural extension of planar map colorings.

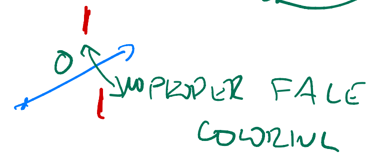
Theorem 7 (Tutte)**.** *A plane graph G is k-face-colorable, if and only if G admits a nowhere-zero k-flow.*

Proof. ($\Rightarrow$). Let λ be a k-face-coloring of G with colors from the set $\{0, 1, \ldots, k-1\}$. Define an orientation D and a weight function f in the following. Let $e = uv \in E(G)$ be an arbitrary edge from G and let F_1 and F_2 be the faces incident with e. Now, orient e in such a way that the face with the bigger color is on its right side, and for its weight just let $f(e) = |\lambda(F_1) - \lambda(F_2)|$.



COLOR

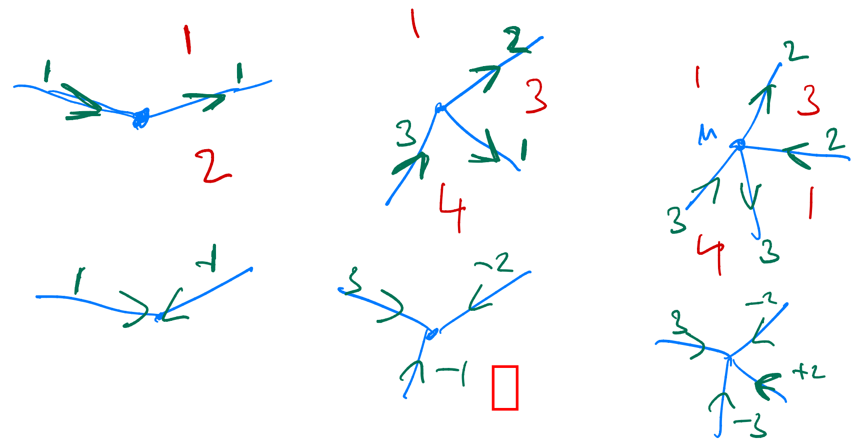




6: Show that (D, f) *is a nowhere-zero k-flow of G.*

1) NOWHERE ZERO

2) KIRCHOFF LAW AT EVERY VERTEX



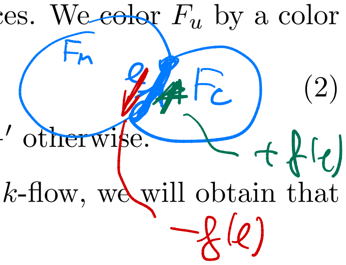
$\sum_i D(u,v_i) \cdot f(u,v_i) =$

$\sum_i \lambda(F_i) - \lambda(F_{i+1}) = 0$

DEGREE OF u

($\Leftarrow$). Suppose G admits a nowhere-zero k-flow (D, f). We will construct a desired coloring $\lambda : F(G) \to \{0, 1, \ldots, k-1\}$ in the following way. First we choose one face and we color it by one of these colors. Next, we repeat the following procedure until all faces are colored: choose one face F_u that is not colored but that has a neighbouring face that is colored, say F_c, and let e be the edge that border both faces. We color F_u by a color $\lambda(F_u)$ so that the following hold:

$$\lambda(F_u) \equiv \lambda(F_c) \pm f(e) \pmod{k} \tag{2}$$



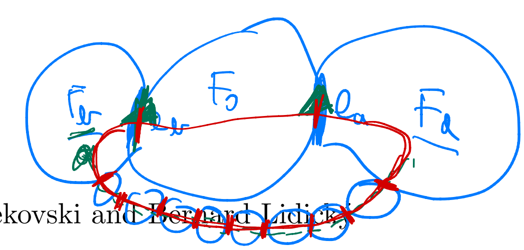
with operation $'+'$, when F_c is on the right side of the edge e and with operation $'-'$ otherwise.

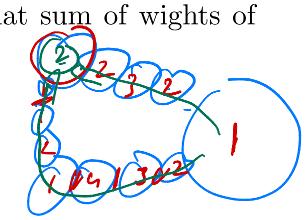
In what follows we will show that λ is well defined. And, since f is a nowhere-zero k-flow, we will obtain that λ is proper k-face-coloring of G.

7: Let F_0 be a non-colored face that is adjacent to two colored faces F_a and F_b and let e_a be an edge between F_0 and F_a, and similarly, e_b be an edge between F_0 in F_b. We may assume that F_0 is on the left side of the edge e_a and on the right side of the the edge e_b. Thus, it will be enough to show that

$$\lambda(F_a) + f(e_a) \equiv \lambda(F_b) - f(e_b) \pmod{k}. \tag{3}$$

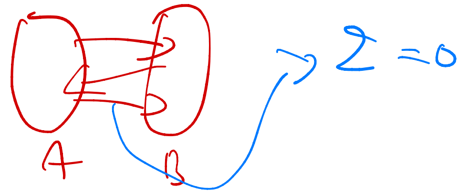
Idea: show that reaching F_b from in another way F_a gives the same color to F_0. Use that sum of wights of edges in a cut is 0.





~ EDGE CUT & SUM ON EDGE CUT IS 0

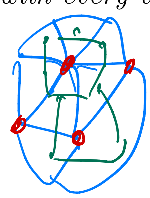
IDEA .. SUM ON CUT = 0 $\Rightarrow$ F_0
GETS SAME COLOR FROM BOTH SIDES



$\square$

Now we derive proof the Heawood theorem about 3-colorings of even triangulation as a side results.

Theorem 8 (Heawood)**.** *A planar triangulation with every vertex of even degree is 3-colorable.*



8: Prove Heawood's theorem.

G

DUAL OF G ?

G^* IS CUBIC GRAPH   BIPARTITE?

$\Uparrow$

ALL FACES IN G^* ARE EVEN

$\Rightarrow$ G^* HAS NOWHERE ZERO 3-FLOW

$\Updownarrow$

FACES OF G^* A 3-COLORABLE

$\Downarrow$

VERTICES OF G ARE 3-COLORABLE

3 Tutte's conjectures

The previous result that dualize the concepts of face-coloring planar graphs and flows on planar graphs, motivated Tutte to state four interesting conjectures. The first two conjectures of Tutte consider the upper bound of the flow number.

k-Flow Conjecture. *There exists an integer k such that every bridgless cubic graph admits nowhere-zero$k$-flow.*

5-Flow Conjecture. *Every graph without bridges admits nowhere-zero 5-flow.*

The first conjecture was independently solved by Kilpatrick and Jaeger. Both of the showed that the upper bound is $k = 8$ of the flow number. Later Seymour proved that 6 is also upper bound, i.e. $\kappa(G) \leq 6$ *for every graph G without bridges.*

The 5-Flow Conjecture is generalization of the 5-Color Theorem and we know that the Petersen graph does not admit nowhere-zero 4-flow. So in this conjecture, we cannot replace 5 by 4 but the next Tutte conjecture consider 4-flows. First note that we can restate the Four Color Theorem as - *Every bridgeless planar graph admits a nowhere-zero 4-flow.* The Tutte guess is that we can go out of planarity with this. Beside the Hadwiger conjecture, it is the strongest generalization of the Four Color Theorem.

4-Flow Conjecture. *Every bridgeless graph that does not contain the Petersen graph as a minor admits a nowhere-zero 4-flow.*

Note that the above conjecture restricted to the cubic graphs is precisely Tutte's about 3-edge-colorings of Petresen-minor-free cubic graphs.

The last Tutte conjecture generalize the Grötzsch theorem. If we dualise this theorem, it says that *every planar graph without 1-edge-cuts and 3-edge-cuts is 3-face-colorable.* And, the Tutte conjecture extends this statement out of the plane.

3-Flow Conjecture. *Every bridgeless graph without 3-edge-cuts admits a nowhere-zero 4-flow.*